12.1 Finding Limits

Who uses this?
Economics, Business and Accounting
Engineering
Natural Sciences
Mathematics
Computer and Information Science

Evaluate the function f(x) = $x^2 - x + 2$ for values of x near 2 but not = to 2.

$$f(3) = 8$$
$$f(2.5) = 5.75$$
$$f(2.25) = 4.8125$$
$$f(2.125) = 4.3906$$
$$f(2.0625) = 4.1914$$
$$f(2.0001) = 4.0003$$

Notice that the closer x gets to 2, the closer f(x) gets to 4…

Definition of the Limit of a Function

$\lim_{x \to a} f(x) = L$

"the limit of f(x) as x approaches a, equals L

if we can make the value of f(x) arbitrarily close to L by taking x to be sufficientlly close to a, but not = to a

- The values of f(x) get closer and closer to the number L as x gets closer and closer to the number a (from either side) but x doesn't = a

Estimate the Limit: $\lim_{x \to 1} \frac{x-1}{x^2-1}$

X < 1	F(x)
.5	0.6667
.9	0.5263
.999	0.5003

X > 1	F(x)
1.5	0.4000
1.1	0.4762
1.001	0.4998

$$\therefore L = 0.5$$

Limits that Fail to Exist

- A function with a jump:

$$H(t) = \begin{cases} 0 & \text{if } t < 0 \\ 1 & \text{if } t \geq 0 \end{cases}$$

- A function with a vertical asymptote:

$$\lim_{x \to 0} \frac{1}{x}$$

These limits fail to exist because the function does not approach the same value for f(x) on both sides of x. (continue for more detail)

One Sided Limits

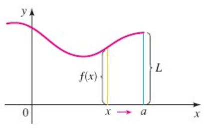
$\lim_{x \to a^-} f(x) = L$

the left hand limit of f(x) as x approaches a or the limit of f(x) as x approaches a from the left = L if we can make the value of f(x) arbitrarily close to L by taking x to be sufficiently close to a and x less than a

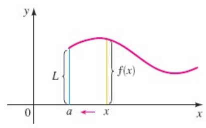
$\lim_{x \to a^-} f(x) = L$

- What is the limit when you approach x from the left?
- What is the limit when you approach x from the right?

$\lim_{x \to a^+} f(x) = L$



(a) $\lim_{x \to a^-} f(x) = L$



(b) $\lim_{x \to a^+} f(x) = L$

- What is the limit when you approach x from the left?
- What is the limit when you approach x from the right?

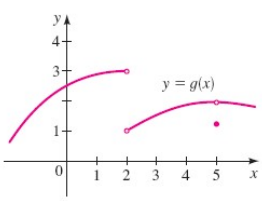
$\lim_{x \to a} f(x) = L$ if and only if $\lim_{x \to a^-} f(x) = L$ and $\lim_{x \to a^+} f(x) = L$

Thus, if the left-hand and right-hand limits are different, the (two-sided) limit does not exist. We use this fact in the next two examples.

The graph of a function g is shown in Figure 1 (if they exist) of the following:

(a) $\lim_{x \to 2^-} g(x)$, $\lim_{x \to 2^+} g(x)$, $\lim_{x \to 2} g(x)$

(b) $\lim_{x \to 5^-} g(x)$, $\lim_{x \to 5^+} g(x)$, $\lim_{x \to 5} g(x)$



$\lim_{x \to 2^-} g(x) = 3$ $\lim_{x \to 5^-} g(x) = 2$

$\lim_{x \to 2^+} g(x) = 1$ $\lim_{x \to 5^+} g(x) = 2$

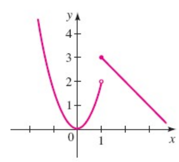
$\lim_{x \to 2} g(x) = DNE$ $\lim_{x \to 5} g(x) = 2$

Let f be the function defined by

$$f(x) = \begin{cases} 2x^2 & \text{if } x < 1 \\ 4 - x & \text{if } x \geq 1 \end{cases}$$

Graph f, and use the graph to find the following:

(a) $\lim_{x \to 1^-} f(x)$ **(b)** $\lim_{x \to 1^+} f(x)$ **(c)** $\lim_{x \to 1} f(x)$



$\lim_{x \to 1^-} f(x) = 2$ $\lim_{x \to 1^+} f(x) = 3$

$\lim_{x \to 1} f(x) = DNE$

12.2 Finding Limits Algebraically

Who uses this?

Actuary

Aerospace Engineer

Earthquake safety Engineer

Limit of a Constant : the limit of a constant function at any point c is the constant value

$$\lim_{x \to a} c = c$$

- Limit of the Identity Function: the limit of the identity function at any point a is a

$$\lim_{x \to a} x = a$$

Limits Laws

Suppose that c is a constant and that the following limits exist:

$$\lim_{x \to a} f(x) \quad \text{and} \quad \lim_{x \to a} g(x)$$

Then

1. $\lim_{x \to a} [f(x) + g(x)] = \lim_{x \to a} f(x) + \lim_{x \to a} g(x)$ — Limit of a Sum
2. $\lim_{x \to a} [f(x) - g(x)] = \lim_{x \to a} f(x) - \lim_{x \to a} g(x)$ — Limit of a Difference
3. $\lim_{x \to a} [cf(x)] = c \lim_{x \to a} f(x)$ — Limit of a Constant Multiple
4. $\lim_{x \to a} [f(x)g(x)] = \lim_{x \to a} f(x) \cdot \lim_{x \to a} g(x)$ — Limit of a Product
5. $\lim_{x \to a} \frac{f(x)}{g(x)} = \frac{\lim_{x \to a} f(x)}{\lim_{x \to a} g(x)}$ if $\lim_{x \to a} g(x) \neq 0$ — Limit of a Quotient

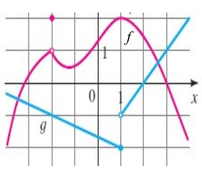
Power and nth root property

$\lim_{x \to a} x^n = a^n$ where n is a positive integer

$\lim_{x \to a} \sqrt[n]{x} = \sqrt[n]{a}$ where n is a positive integer

and a > 0

Use the limit laws and the graphs of f and g to evaluate the following:



(a) $\lim_{x \to -2} [f(x) + 5g(x)]$ (b) $\lim_{x \to 1} [f(x)g(x)]$

(c) $\lim_{x \to 2} \frac{f(x)}{g(x)}$ (d) $\lim_{x \to 1} [f(x)]^3$

$a.)\, 1 + 5(-1)$
-4

$b.)\, (2)(DNE)$
DNE

$c.)\, \frac{1.5}{0}$
DNE

$d.)\, (2)^3$
8

Evaluate the following limits:

$$\lim_{x \to -2} (6x^2 - 3x + 1)$$

$\lim_{x \to -2} (6x^2) - \lim_{x \to -2} (3x) + \lim_{x \to -2} (1)$

$6 \lim_{x \to -2} (x^2) - 3 \lim_{x \to -2} (x) + 1$

$6(-2)^2 - 3(-2) + 1 \quad = 24 + 6 + 1 \quad = 31$

$$\lim_{x \to -2} \frac{x^3 + 2x^2 - 1}{5 - 3x} \quad \frac{\lim_{x \to -2} (x^3 + 2x^2 - 1)}{\lim_{x \to -2} (5 - 3x)} \quad = \frac{-1}{11}$$

$$\frac{\lim_{x \to -2} (x^3) + 2 \lim_{x \to -2} (x^2) - \lim_{x \to -2} (1)}{\lim_{x \to -2} (5) - 3 \lim_{x \to -2} (x)} = \frac{(-2)^3 + 2(-2)^2 - 1}{5 - 3(-2)} = \frac{-8 + 8 - 1}{5 + 6}$$

Evaluate the following limits:

$$\lim_{x \to 3} \sqrt{8 - x}$$

$\sqrt{\lim_{x \to 3} (8 - x)}$

$\sqrt{\lim_{x \to 3} 8 - \lim_{x \to 3} x}$

$\sqrt{8 - 3}$

$\sqrt{5}$

Limits by Direct Substitution

- If f is a polynomial or a rational function and a is in the domain of f then:

$$\lim_{x \to a} f(x) = f(a)$$

$$\lim_{x\to -1}\left(-3x^4+5x^3-2x^2+x+4\right)$$

$L=-7$

$$\lim_{x\to 2}\frac{x^2+6x}{x^4+3} \quad \frac{(2)^2+6(2)}{(2)^4+3}$$

$$\frac{4+12}{16+3}=\frac{16}{19}$$

Find the limit:

$$\lim_{x\to 1}\frac{x^2-1}{x-1} \qquad x\neq 1$$

$$\lim_{x\to 1}\frac{(x-1)(x+1)}{x-1}$$

$$\lim_{x\to 1}\frac{(x+1)}{1}$$

$$\frac{(1+1)}{1}=2$$

Find limits by factoring

$$\lim_{x\to 4}\frac{x-4}{x^2-16}$$

$$\lim_{x\to 4}\frac{x-4}{(x+4)(x-4)}$$

$$\lim_{x\to 4}\frac{1}{(x+4)}=\frac{1}{(4+4)}=\frac{1}{8}$$

$$\lim_{x\to 3}\frac{x-3}{x^3-3x^2-7x+21} \quad \lim_{x\to 3}\frac{x-3}{(x^2-7)(x-3)}$$

$$\lim_{x\to 3}\frac{1}{(x^2-7)}=\frac{1}{(9-7)}=\frac{1}{2}$$

Find limits by simplifying

$$\lim_{h\to 0}\frac{(4+h)^2-16}{h}$$

$$\lim_{h\to 0}\frac{16+8h+h^2-16}{h}$$

$$\lim_{h\to 0}\frac{8h+h^2}{h}$$

$$\lim_{h\to 0}(8+h)=8$$

Find limits by rationalizing

$$\lim_{x\to 1}\frac{\sqrt{x}-1}{x-1}\cdot\frac{\sqrt{x}+1}{\sqrt{x}+1}$$

$$\lim_{x\to 1}\frac{x-1}{(x-1)\left(\sqrt{x}+1\right)}$$

$$\lim_{x\to 1}\frac{1}{\left(\sqrt{x}+1\right)}=\frac{1}{(1+1)}=\frac{1}{2}$$

$$\lim_{x\to 9}\frac{\sqrt{x}-3}{x-9}\cdot\frac{\sqrt{x}+3}{\sqrt{x}+3} \quad \lim_{x\to 9}\frac{x-9}{(x-9)\left(\sqrt{x}+3\right)}$$

$$\lim_{x\to 9}\frac{1}{\left(\sqrt{x}+3\right)}=\frac{1}{(3+3)}=\frac{1}{6}$$

WS 12.1-12.2

Homework